PROPAGATION OF FINITE-AMPLITUDE LONG-WAVE DISTURBANCES IN AEROSOLS

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UDC 534.222

The propagation and structure of a pressure (density, velocity, or temperature) disturbance in a volume containing solid particles are of considerable theoretical and practical significance. Of particular interest is the structure of a transient disturbance, since the dimensions of engineering equipment utilizing heterogeneous media are such that a steady-state structure cannot evolve. Typical of such equipment are solid-propellant rocket engines and nuclear reactors using a disperse heat-transfer agent.

Davidson [1] gives an equation describing the evolution of finite-amplitude waves in a medium containing an aerosol. This equation has the form of the Burgers equation augmented with integral terms to account for mechanical and thermal relaxation processes. Interphase heat transfer is taken into account by Newton's law, but the latter represents a crude approximation for highly transient states.

In this study particular attention is given to the derivation and analysis of equations for the evolution of finite-amplitude disturbances when the system involves a single relaxation process associated with transient heat transfer between the aerosol particles and the gas.

The volume fraction of solid particles is assumed to be so small that interaction between individual particles can be neglected. This restriction permits the detailed heat-transfer processes to be analyzed independently of the overall dynamical problem.

We assume that the wavelength spans a sufficient number of particles for the mixture to be treated as a continuum. The disturbances are assumed to be plane and have a long wavelength. Under the stated assumptions the propagation of disturbances can be investigated within the contact of the homogeneous model [2].

We consider the evolution of disturbances in an aerosol containing solid particles of equal radius δ, density ρ_p, and temperature Θ in the approximation of the single-velocity model. The number of particles m is assumed to be constant in unit volume, and their volume so small as to be negligible in the conservation equations in comparison with the volume occupied by the gas [2].

Neglecting viscosity and heat conduction in the conservation equations and retaining only thermal interaction between phases, we have the following system of equations for describing the process:

$$\begin{gathered}\rho_t + (u\rho)_x = 0,\ \rho u_t + \rho u u_x + R(\rho\vartheta)_x = 0,\\ \rho c_V \vartheta_t + \rho c_V u \vartheta_x + R\rho\vartheta u_x = Q,\ \Theta_t - a\Theta_{rr} - (2a/r)\Theta_r = 0.\end{gathered} \tag{1}$$

The first three equations are the equations of continuity, motion, and energy for an ideal gas, where ρ, u, and ϑ are the density, velocity, and temperature of the gas; Q is the total heat flux from the gas to the particle cloud; R is the universal gas constant; and c_V is the isochoric specific heat. The fourth equation of the system (1) is used to determine the thermal interaction between a particle and the gas, where a is the thermal diffusivity and r is the radial distance. This equation is solved subject to the boundary conditions

$$\begin{gathered}t = 0,\ \Theta = \Theta_0 = 0,\\ r = \delta \text{ for } t > 0,\ \Theta = \Theta_\delta = \vartheta(t),\ \partial\Theta/\partial r|_{r=0} = 0.\end{gathered} \tag{2}$$

It can also be solved under Cauchy boundary conditions.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 33-38, September-October, 1980. Original article submitted March 10, 1980.

 0021-8944/80/2105-0602$07.50

We analyze the solution of the particle heat-conduction equation under boundary conditions (2). It has the form [3]

$$\Theta=\sum_{n=1}^{\infty}\frac{2\delta}{r_n\pi}(-1)^{n+1}\sin\left(n\pi\frac{r}{\delta}\right)\frac{\partial}{\partial t}\int_0^t\vartheta(\tau)\,e^{-\frac{n^2\pi^2a}{\delta^2}(t-\tau)}d\tau.$$

The heat flux toward a particle is written in the form

$$q_p=\lambda_p\left(\frac{\partial\Theta}{\partial r}\right)_{r=\delta}=-\frac{2\lambda_p}{\delta}\sum_{n=1}^{\infty}\frac{\partial}{\partial t}\int_0^t\vartheta(\tau)\,e^{-\frac{n^2\pi^2a}{\delta^2}(t-\tau)}d\tau,$$

where λ_p is the particle thermal conductivity. In the case $n^2\pi^2a(t-\tau)/\delta^2 \gg 1$ the long-wave (low-frequency) limit is realized, such that $\vartheta(\tau)$ is a slightly varying function and can be expanded into a series in a neighborhood of the point t with respect to a small time lag $(t-\tau)$:

$$\vartheta(\tau)=\vartheta(t)-\frac{\partial\vartheta}{\partial t}(t-\tau)+\frac{\partial^2\vartheta}{\partial t^2}(t-\tau)^2-\ldots$$

Retaining only terms through the first order, we substitute the first two terms of the series into the integrand. Using the fact that $n^2\pi^2a(t-\tau)/\delta^2\to\infty\exp[-n^2\pi^2a(t-\tau)/\delta^2]\to 0$ [already for $n^2\pi^2a(t-\tau)/\delta^2 > 4.6$ the function $\exp[-n^2\pi^2a(t-\tau)/\delta^2] < 0.01$], we finally obtain

$$\int_0^t\vartheta(\tau)\,e^{-\frac{n^2\pi^2a}{\delta^2}(t-\tau)}d\tau=K\vartheta(t)-L\frac{\partial\vartheta}{\partial t},$$

where $K=\delta^2/n^2\pi^2a$; $L=\delta^4/n^4\pi^4a^2$, and the heat flux

$$q_p=\sum_{n=1}^{\infty}-2\frac{\lambda_p}{\delta}\left[K\frac{\partial\vartheta}{\partial t}-L\frac{\partial^2\vartheta}{\partial t^2}\right]=-A'\rho_pc_p\delta\frac{\partial\vartheta}{\partial t}+B'\rho_pc_p\delta\frac{\delta^2}{a}\frac{\partial^2\vartheta}{\partial t^2}, \tag{3}$$

where c_p is the specific heat of the particle material and

$$A'=\sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}=\frac{1}{3};\quad B'=\sum_{n=1}^{\infty}\frac{2}{n^4\pi^4}=\frac{1}{45}.$$

We use the expression (3) derived above to write the energy equation in the form

$$\begin{gathered}\rho\vartheta_t+\rho u\vartheta_x+\rho\frac{R}{c_V}\vartheta u_x=-A\rho_p\vartheta_t+\tau_p'\rho_p\vartheta_{tt},\\ A=M\frac{c_p}{c_V},\quad \tau_p'=M\frac{c_p}{c_V}\frac{\delta^2}{15a},\quad M=\frac{4}{3}\pi\delta^3m.\end{gathered} \tag{4}$$

Now our process is described by the first and second equations of the system (1) and Eq. (4).

We consider the propagation of infinitesimally small disturbances in such a system. We assume that the relative deviations of the density, velocity, and temperature from their equilibrium values $[(\rho-\rho_0)/\rho_0,\ u/c_\infty,\ (\vartheta-\vartheta_0)/\vartheta_0]$ are first-order quantities. Substituting $\rho=\rho_0+\rho'$, $u=u'$, $\vartheta=\vartheta_0+\vartheta'$ into the system of equations derived above and discarding second- and higher-order terms, we obtain

$$\begin{gathered}\rho_t'+\rho_0u_x'=0,\quad \rho_0u_t'+\rho_0R\vartheta_x'+R\vartheta_0\rho_x'=0,\\ \vartheta_t'(\rho_0+A\rho_p)+\rho_0(\varkappa-1)\vartheta_0u_x'-\tau_p'\rho_p\vartheta_{tt}'=0.\end{gathered} \tag{5}$$

The system (5) is the initial system for the derivation of one wave equation describing the propagation of sound waves in the given media. We seek a solution of this system in traveling-wave form $(x>0)\,\rho'\sim\rho_0\exp[i(kx-\omega t)]$, $u'\sim c_0\exp[i(kx-\omega t)]$, $\vartheta'\sim\vartheta_0\exp[i(kx-\omega t)]$, where a functional relationship exists between the wave number k and the frequency ω, and k is a complex quantity in general. This relationship is determined from the compatibility condition for the solutions of (5), which yields the equations

$$\begin{gathered}\tau_p\frac{\partial}{\partial t}(u_{tt}-c_\infty^2u_{xx})-Du_{tt}+c_0^2u_{xx}=0,\\ \tau_p\omega^3-iD\omega^2-k^2\omega\tau_pc_\infty^2+ik^2c_0=0,\end{gathered} \tag{6}$$

in which

$$D=\frac{\rho_0+A\rho_p}{\rho_0};\quad R\vartheta_0=c_\infty^2;\quad R\vartheta_0\ \frac{\varkappa\rho_0+A\rho_p}{\rho_0}=c_0^2;\quad \tau_p=\tau_p'\rho_p/\rho_0.$$

For the phase velocity $V_p = \omega/k$ we obtain from (6)

$$V_p=\left[(c_\infty^2\omega\tau_p-ic_0^2)/(\omega\tau_p-iD)\right]^{1/2}. \tag{7}$$

Separating real and imaginary parts V_{pR} and V_{pIm} in the latter expression, we have

$$V_{p\mathrm{Im}}^2=-\frac{Dc_0^2+\omega^2\tau_p^2c_\infty^2}{2\left(\omega^2\tau_p^2+D^2\right)}+\sqrt{\frac{\left(\omega^2\tau_p^2c_\infty^4+c_0^4\right)\left(\omega^2\tau_p^2+D^2\right)}{4\left(\omega^2\tau_p^2+D^2\right)^2}}; \tag{8}$$

$$V_{p\mathrm{R}}^2=\frac{\omega^2\tau_p^2\left(Dc_\infty^2-c_0^2\right)^2}{2\left(\omega^2\tau_p^2+D^2\right)\left[-\left(Dc_0^2+\omega^2\tau_p^2c_\infty^2\right)+\sqrt{\left(Dc_0^2+\omega^2\tau_p^2c_\infty^2\right)^2+\left(\omega\tau_pDc_\infty^2-\omega\tau_pc_0^2\right)^2}\right]}. \tag{9}$$

These equations are valid for small values of $\omega\tau_p$, since we have imposed beforehand the constraint $\omega\tau_p < 1$. Expanding the phase velocity into a series about $\omega\tau_p = 0$ and retaining only terms through the first order with respect to $\omega\tau_p$, we have

$$V_p=V_p(\omega\tau_p=0)+\left.\frac{\partial V_p}{\partial(\omega\tau_p)}\right|_{\omega\tau_p=0}\omega\tau_p+O\left[(\omega\tau_p)^2\right].$$

Calculating the coefficients of the powers of $\omega\tau_p$ by means of Eq. (7), we obtain

$$V_p=\frac{c_0}{\sqrt{D}}-i\frac{c_0^2-Dc_\infty^2}{Dc_0^2}\frac{c_0}{\sqrt{D}}\frac{\omega\tau_p}{2}+O\left[(\omega\tau_p)^2\right],$$

where the real and imaginary dispersions are

$$V_{p\mathrm{R}}=\frac{c_0}{\sqrt{D}}; \tag{10}$$

$$V_{p\mathrm{Im}}=-\frac{c_0^2-Dc_\infty^2}{Dc_0^2}\frac{c_0}{\sqrt{D}}\frac{\omega\tau_p}{2}. \tag{11}$$

The asymptotic relations (10) and (11) well describe the behavior of the total dispersion curves (8) and (9) in the domain

$$0<\omega\tau_p<\left[\frac{c_0^2-Dc_\infty^2}{Dc_0^2}\right]^{1/2}.$$

The dispersion curves for the system of equations (5) in the case of copper particles with a radius of 30 μ and weight concentration of 0.1 are given in Fig. 1, in which curves 1 and 2 represent the asymptotic behavior of the imaginary and real parts, respectively:

$$|\overline{V}_{p\mathrm{Im}}|=|V_{p\mathrm{Im}}|/\max|V_{p\mathrm{Im}}|.$$

We consider the finite-amplitude waves traveling in the positive direction. We transform to an accompanying coordinate system, $\tau = t - x/c_\infty$. Inasmuch as the wave amplitudes are small, the distortions of the wave profile due to dissipation and nonlinearity will also be small at distances of wavelength order, and the process must be described by a function of the form $\Phi(\mu x, \tau)$, where μ is a first-order quantity. In accordance with the substitution we have

$$\frac{\partial}{\partial\tau}=\frac{\partial}{\partial t},\quad \frac{\partial}{\partial x}=-\frac{1}{c_\infty}\frac{\partial}{\partial\tau}+\mu\frac{\partial}{\partial x}.$$

Now the system comprising the first two equations (1) and Eq. (4) are written as follows in the variables τ and x, correct to within second-order accuracy:

$$\frac{\partial\rho'}{\partial\tau}-\frac{\rho_0}{c_\infty}\frac{\partial u'}{\partial\tau}+\mu\rho_0\frac{\partial u'}{\partial x}-\frac{\rho'}{c_\infty}\frac{\partial u'}{\partial\tau}-\frac{u'}{c_\infty}\frac{\partial\rho'}{\partial\tau}=0,$$

$$\rho_0\frac{\partial u'}{\partial\tau}+\rho'\frac{\partial u'}{\partial\tau}-\frac{\rho_0R}{c_\infty}\frac{\partial\vartheta'}{\partial\tau}+\mu\rho_0R\frac{\partial\vartheta'}{\partial x}-\frac{\rho_0u'}{c_\infty}\frac{\partial u'}{\partial\tau}-\frac{R\rho'}{c_\infty}\frac{\partial\vartheta'}{\partial\tau}-\frac{R\vartheta_0}{c_\infty}\frac{\partial\rho'}{\partial\tau}+\mu R\vartheta_0\frac{\partial\rho'}{\partial x}-\frac{R\vartheta'}{c_\infty}\frac{\partial\rho'}{\partial\tau}=0, \tag{12}$$

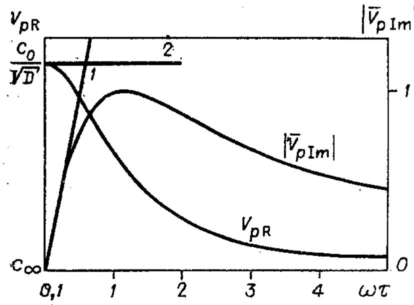

Fig. 1

$$\rho_0\frac{\partial\vartheta'}{\partial\tau}+\rho'\frac{\partial\vartheta'}{\partial\tau}-\frac{\rho_0u'}{c_\infty}\frac{\partial\vartheta'}{\partial\tau}-\frac{\rho_0\vartheta_0(\varkappa-1)}{c_\infty}\frac{\partial u'}{\partial\tau}+\mu\rho_0\vartheta_0(\varkappa-1)\frac{\partial u'}{\partial\tau}$$
$$-\frac{\rho'\vartheta_0(\varkappa-1)}{c_\infty}\frac{\partial u'}{\partial\tau}-\frac{\rho_0\vartheta'(\varkappa-1)}{c_\infty}\frac{\partial u'}{\partial\tau}+A\rho_p\frac{\partial\vartheta'}{\partial\tau}-\tau_p\rho_p\frac{\partial^2\vartheta'}{\partial\tau^2}=0.$$

It is well known [4] that a single partial differential equation of order k can be reduced to a system of k first-order equations. Analogously, a system of k first-order partial differential equations can sometimes be reduced by differentiation to a single equation of order k. One equation, of course, is more readily amenable to physical analysis, and to obtain a single equation is often the sole objective of many papers. The remarkable thing in this approach is the fact that the equation so obtained sometimes proves to be already well known and has been analyzed in detail.

The system of equations (12) is reducible to a single equation by analogy with [5]. For this purpose we multiply the first equation of the system by $1/\rho_0$, the second by $1/\rho_0c_\infty$, and the third by $1/\rho_0\vartheta_0$. Adding the results and replacing ρ'/ρ_0 by u'/c_∞ as well as ϑ'/ϑ_0 by $(\varkappa-1)u'/c_\infty$ in all second-order terms, we arrive at the Burgers equation

$$2\mu\varkappa\frac{\partial u'}{\partial x}-\frac{u'}{c_\infty^2}\frac{\partial u'}{\partial\tau}(\varkappa^2+2)-\frac{\partial u'}{\partial\tau}\frac{1}{c_\infty}\left(1-A\frac{\rho_p}{\rho_0}\right)(\varkappa-1)-\frac{\tau_p\rho_p}{\rho_0}\frac{(\varkappa-1)}{c_\infty}\frac{\partial^2u'}{\partial\tau^2}=0$$

or, in dimensionless form with the inclusion of 2μ in x, we have

$$\frac{\partial u}{\partial x}-u\frac{\partial u}{\partial\tau}\frac{(\varkappa^2+2)}{\varkappa}-\frac{\partial u}{\partial\tau}\left(1-A\frac{\rho_p}{\rho_0}\right)\frac{\varkappa-1}{\varkappa}-\frac{\tau_p}{T}\frac{\rho_p}{\rho_0}\frac{\varkappa-1}{\varkappa}\frac{\partial^2u}{\partial\tau^2}=0, \quad (13)$$

where T is the wave period. The coefficient in front of $\partial^2u/\partial\tau^2$ is the viscosity of the aerosol

$$\nu_c=c_\infty^2\frac{\delta^2}{15a}\frac{c_p}{c_V}\frac{\rho_p}{\rho_0}M$$

in dimensioned form. The reverse substitution $u'/c_\infty=\rho'/\rho_0$ and $\vartheta'/\vartheta_0=(\varkappa-1)\rho'/\rho_0$ or $u'/c_\infty=\vartheta'/(\varkappa-1)\vartheta_0$ and $\rho'/\rho_0=\vartheta'/(\varkappa-1)\vartheta_0$ leads to the same equation for the increments of the density ρ' and temperature ϑ'. The same approach with the inclusion of heat transfer between a gas bubble and a surrounding liquid during transmission of a wave disturbance has been used in [6].

Thus, the propagation of long-wave disturbances of finite amplitude in aerosols is described by the Burgers equation with coefficients that incorporate information on the individual characteristics of the particles. It has been shown that transient heat transfer between the particles and the gas in the wave induces a relaxational viscosity. This effect has been deduced for the first time. The solution of this equation has been investigated in [5, 7]; in particular, the familiar Hopf substitution can be used to reduce Eq. (13) to the linear heat-conduction equation. It qualitatively describes the evolution of a shock wave in which nonlinearity is equalized by relaxational viscosity, which is induced here by transient interphase heat transfer. Finite disturbances are attenuated with time until their amplitude diminishes to zero.

It can be shown that when longitudinal heat conduction and shear viscosity are included in the initial system of equations, they add linearly to the resulting relaxational viscosity.

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THERMOCAPILLARY MOTION IN A GAS–LIQUID MIXTURE

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1. Equation of Motion. Let a viscous incompressible liquid with gas bubbles be found in the region of space Ω. The number of bubbles is sufficiently large that a number $a \ll d$ can be found, where d is the diameter of Ω, such that any sphere of radius a belonging to Ω contains a number of bubbles $N \gg 1$. The bubbles are assumed to be spheres of identical radius R. If the characteristic distance between bubble centers l is sufficiently small in comparison with the characteristic distance L over which the mean mixture parameters vary, the concepts of mechanics of heterogeneous media (see, e.g., [1]) are valid.

If the external mass forces are small and the acceleration of the liquid is also small, the main source of motion is the inhomogeneity of the temperature field in the liquid and the thermocapillary effect induced by it [2].

We denote by c the bulk concentration of the gas, by **u** and **v** the average velocities of the gas and liquid phases, respectively, and by T the temperature. The exact continuity equation (within the scope of fluid mechanics) for the liquid phase is

$$\partial(1-c)/\partial t + \operatorname{div}[(1-c)\mathbf{v}] = 0. \tag{1.1}$$

Allowing the gas density to satisfy $\rho_g = \text{const}$, the continuity equation for the gas phase is similar to (1.1):

$$\partial c/\partial t + \operatorname{div}(c\mathbf{u}) = 0. \tag{1.2}$$

The possible gas-exchange process between bubbles and the liquid due to diffusion processes is not taken into account. For simplicity, we do not take into account either the more important process of bubble coagulation, which up to a certain extent is justified in the case of a dilute system.

Taking into account that the shear viscosity of a suspension of gas bubbles equals $(1 + c)\nu$, where ν is the viscosity of the liquid, and neglecting quadratic terms of order c^2 in the viscous stresses, one can write the momentum equation of the liquid in the form

$$(1-c)d\mathbf{v}/dt = -\rho^{-1}\nabla p + (1-c)\mathbf{g} + 2\operatorname{div}[(1+c)\nu S], \tag{1.3}$$

where S is the velocity deformation tensor; p, pressure; and **g**, acceleration of the external mass forces. Equation (1.3) is valid for small Reynolds numbers of bubble flow and for sufficiently large characteristic times of motion, when effects of associated bubble masses can be neglected.

Moscow, Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 38-45, September-October, 1980. Original article submitted March 13, 1980.

 0021-8944/80/2105-0606$07.50